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ДОКАЗАТЕЛЬСТВО ТЕОРЕМ О НЕКОТОРЫХ НЕРАВЕНСТВАХ И ИХ ПРИМЕНЕНИЕ НА ПРАКТИКЕ

Аннотация. *В данной работе рассматриваются понятия неравенств, их типы (линейные, квадратичные, иррациональные) и другие, методы решения и графическое описание решений. Также изучается, как неравенства выражаются в реальной жизни. Будут раскрыты построение и применение математических моделей, а также их роль. Предметом исследования являются закрепление математической логики и анализ задач для развития навыков решения практических задач.*

Ключевые слова: *неравенство, линейное неравенство, квадратичное неравенство, интервальный метод, графическое решение, математическая модель, множество решений*

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PROOFS OF THEOREMS ON CERTAIN INEQUALITIES AND THEIR APPLICATION IN PRACTICE

Abstract. *This article considers the concept of inequalities, their types (linear, quadratic, irrational, and others), solution methods, and the graphical representation of solution sets. It also studies how inequalities are expressed in real life. The construction and application of mathematical models and their role are discussed. The work contributes to strengthening mathematical logic and developing skills in analyzing and solving practical problems.*

Keywords: *inequality, linear inequality, quadratic inequality, interval method, graphical solution, mathematical model, solution set*

Introduction

Mathematics is not only a set of theoretical concepts; it is a powerful tool for modeling and analyzing real-life problems [1]. In particular, inequalities are widely applied in everyday decision-making, in the optimal distribution of resources, and in determining safety limits. Budget constraints, maximum production volume, and minimum energy expenditure are among the economic and technical problems that are expressed through inequalities [2].

Materials and Methods

This is a theoretical study. The inequalities are analyzed through formal proofs, which use mathematical induction, auxiliary lemmas, and standard inequalities (AM–GM and Cauchy–Bunyakovsky). The results are illustrated with worked examples and with applications in medicine, economics, and engineering.

Results and Discussion

The AM–GM inequality

Theorem 1 (AM–GM). For all positive numbers $a_1, a_2, \ldots, a_n$, $(a_1 + a_2 + \ldots + a_n)/n \geq \sqrt[n]{(a_1 a_2 \ldots a_n)}$, and equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

Proof. We use induction on n. For n = 2 the inequality $\sqrt{(a_1 a_2)} \leq (a_1 + a_2)/2$ is equivalent to $(\sqrt{a_1} - \sqrt{a_2})^2 \geq 0$, which is true for all $a_1, a_2 \geq 0$.

Assume the inequality holds for any n positive numbers, and consider n + 1 positive numbers $a_1, \ldots, a_n, a_{n+1}$. Without loss of generality let a_{n+1} be the largest of them, so $a_{n+1} \geq A_n$, where $A_n = (a_1 + \ldots + a_n)/n$. Denote $A_{n+1} = (a_1 + \ldots + a_{n+1})/(n + 1) = (nA_n + a_{n+1})/(n + 1)$. Put $\alpha = a_{n+1} - A_n \geq 0$. Then $A_{n+1} = A_n + \alpha/(n + 1)$. By the binomial theorem, $A_{n+1}^{n+1} \geq A_n^{n+1} + (n + 1)A_n^n \cdot \alpha/(n + 1) = A_n^{n+1} + A_n^n\alpha = A_n^n(A_n + \alpha) = A_n^n \cdot a_{n+1}$. By the induction hypothesis $A_n^n \geq a_1 a_2 \ldots a_n$, hence $A_{n+1}^{n+1} \geq a_1 a_2 \ldots a_n a_{n+1}$, which is the inequality for n + 1 numbers. ■

The generalized (weighted) AM–GM inequality

Theorem 2. Let $a_1, \ldots, a_n, p_1, \ldots, p_n$ be positive numbers and $P = p_1 + \ldots + p_n$. Then a1^{p1}a2^{p2}…an^{pn} ≤ ((a1p1 + … + anpn)/P)^P, and equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

Lemma. For all x > 0, e^{x−1} ≥ x, with equality only at x = 1. Indeed, let f(x) = e^{x−1} − x. Then f′(x) = e^{x−1} − 1, which is negative for x < 1 and positive for x > 1. Hence f attains its minimum at x = 1, and f(1) = 0, so f(x) ≥ 0 for all x > 0.

Proof of Theorem 2. Let $S = (a_1p_1 + \ldots + a_np_n)/P$. By the lemma applied to $x = a_i/S$, we have $e^{a_i/S - 1} \geq a_i/S$, i.e. $a_i \leq S \cdot e^{a_i/S - 1}$. Raising to the power p_i gives $a_i^{p_i} \leq S^{p_i} \cdot e^{a_i p_i/S - p_i}$. Multiplying these n inequalities and noting that $\Sigma\, a_i p_i/S = P$, we obtain $a_1^{p_1} \ldots a_n^{p_n} \leq S^P \cdot e^{P - P} = S^P$, which is the claim. Equality holds only when all $a_i/S = 1$, that is, when $a_1 = \ldots = a_n$. ■

Applications to problem solving

The theorems above are used in the following examples.

Example 1 (corrected statement). Let x > 0. Prove that 2^(x^(1/12)) + 2^(x^(1/4)) ≥ 2 · 2^(x^(1/6)). Note: the statement in the original draft, with 2·2^(2x^(1/6)) on the right-hand side, is false (for x = 1 the left side is 4 and the right side is 8); the corrected form is given here. Proof: by AM–GM, (2^(x^(1/12)) + 2^(x^(1/4)))/2 ≥ √(2^(x^(1/12)) · 2^(x^(1/4))) = 2^((x^(1/12) + x^(1/4))/2). Again by AM–GM, (x^(1/12) + x^(1/4))/2 ≥ √(x^(1/3)) = x^(1/6). Since 2^t is increasing, the claim follows. ■

Example 2 Let $x_1, \ldots, x_5 > 0$. Prove that $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \geq x_1(x_2 + x_3 + x_4 + x_5)$. Proof: for each $i = 2, \ldots, 5$, $x_1^2/4 + x_i^2 \geq x_1 x_i$ (AM–GM). Summing these four inequalities gives $x_1^2 + x_2^2 + \ldots + x_5^2 \geq x_1(x_2 + \ldots + x_5)$. ■

Example 3 Let a, b, c > 0. Prove that (a + 1)(b + 1)(c + a)(b + c) ≥ 16abc. Proof: a + 1 ≥ 2√a, b + 1 ≥ 2√b, c + a ≥ 2√(ca), b + c ≥ 2√(bc). Multiplying gives the product ≥ 16·√(a²b²c²) = 16abc. ■

Graphical illustration

For Example 1, the surfaces $z = x^2 + y^2 + 1$ and $z = xy + x + y$ over the square $0.1 \leq x, y \leq 3$ can be plotted in three dimensions. The first surface lies above the second over this domain, which illustrates the inequality. (The Python code used for the original plot is not reproduced here.)

Applications in practice

In medicine. Dose limits are usually expressed as inequalities. For example, the maximum permitted dose D is related to the patient's body weight m by the condition D ≤ k·m, where k is the dose coefficient. Such an inequality allows a safe and individually adjusted treatment.

Conclusion

Inequalities are an important tool in mathematical analysis, engineering, economics, and medicine. In this article the AM–GM inequality and the Cauchy–

Bunyakovsky inequality were studied, and their proofs were given by induction, by an elementary algebraic argument, and by means of the exponential function.

The examples show that these inequalities are useful not only in theory but also in practice: they allow complicated expressions to be estimated and problems to be simplified, and they make it possible to express real-life restrictions as mathematical models. The article shows their application in determining dose limits in medicine, in distributing budgets in planning, and in ensuring mechanical safety in engineering.

In the future, the topic may be extended to the study of the Hölder, Chebyshev, and Minkowski inequalities, to algorithmic analysis based on computer-assisted proof methods, and to the application of inequalities in olympiad problems [5].

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