METHODOLOGY OF USING SECOND-ORDER DIFFERENTIAL EQUATIONS IN SOLVING PHYSICS PROBLEMS

Buriyev Sardor Bekmurod ugli. Physics teacher at Shakhrisabz Military Academic Lyceum "Temurbeklar maktabi", Emergency Situations of the Republic of Uzbekistan, Kashkadarya, Uzbekistan.

Kayumov Khafiz Asliddin ugli. Physics teacher at Shakhrisabz Military Academic Lyceum "Temurbeklar maktabi", Emergency Situations of the Republic of Uzbekistan, Kashkadarya, Uzbekistan.

Annotation. This article presents a methodology for using second-order homogeneous differential equations in solving physics problems. Physical problems related to mechanics, mechanical vibrations, and electrodynamics are modeled on the basis of differential equations, and the stages of expressing physical processes in the form of equations, as well as methods for finding general and particular solutions, are explained.

Key words: differential equation, characteristic equation, quadratic equation, derivative, stiffness, resistance force, Kirchhoff law, electromotive force, electric circuit.

Introduction. In many branches of modern physics, physical processes are analyzed through mathematical modeling. Second-order differential equations are widely used in mathematical modeling. For example, second-order differential equations play an important role in studying the motion of bodies, understanding processes in electrical circuits, analyzing mechanical and electromagnetic vibrations, and constructing heat dissipation equations. The application of second-order differential equations remains essential in engineering, aerodynamics, microelectronics, and other technological fields, as well as in better understanding processes in nature and achieving new scientific advancements.

A differential equation is an equation that consists of a relationship between an independent variable, an unknown function, and the derivatives of that function:

$$F(x, y, y', y'', ..., y^{(n)}) = 0 \quad (1)$$

Here, *F(x)* is a continuous function of its arguments in the domain under consideration. If the equation has the following structure, it is called a second-order, linear, homogeneous differential equation with constant coefficients:

$$y'' + ay' + by = 0 \quad (1)'$$

Here, *a* and *b* are constant coefficients, and *y=y(x)* is a continuous function of its arguments in the domain under consideration. Methods for solving this equation are presented in the literature [1].

In this article, the application of second-order homogeneous differential equations to solving physics problems, the explanation of methods for solving them, and the clarification of their theoretical and practical significance are aimed. Below, we examine examples related to this topic.

Problem 1. A load with a mass of 5 kg is suspended from a spring with stiffness 500 N/m and oscillates with an amplitude of 5 cm. Ignoring the mass of the spring and air resistance, determine the equations of displacement, velocity, and acceleration of the load.

Solution: The forces acting on the load are shown in figure 1. Under the influence of the load's weight, the spring stretches by x_0. In that case, $mg=kx_0$. Now, we pull the load slightly and release it. If the spring is stretched by Δl relative to its initial position when pulled, then according to Newton's second law, the equation of motion is as follows:

$$ma = mg - k\Delta l \quad (2)$$

Using the equation $mg=ky_0$ above, equation (2) can be rewritten as follows:

$$ma = ky_0 - k\Delta l \quad (3)$$

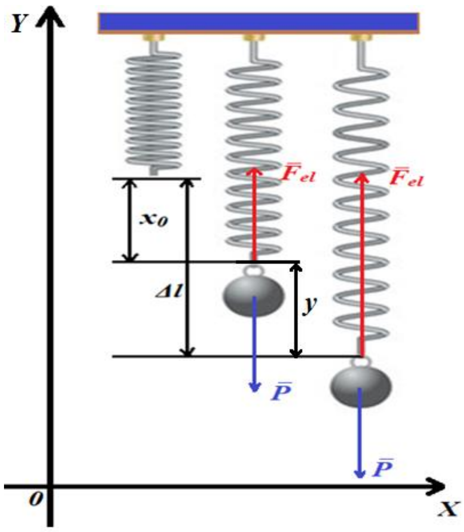

Fig1.Vertical vibration of the body

As can be seen from Fig.1, the coordinate of the load changes along the *Y*-axis. Therefore, the acceleration is the second derivative of the coordinate with respect to time, that is,. $a = \frac{d^2y}{dt^2} = \ddot{y}$. Substituting this expression into equation (3), we obtain the following:

$$m\frac{d^2y}{dt^2} = -k(\Delta l - y_0) \qquad (4)$$

The elongation *Δl* of the spring changes over time. Therefore, the coordinate of the load also changes with time. In equation (4), we set $\Delta l - y_0 = y$ and this will be the coordinate of the load. Using the above relations, we obtain the following differential equation: $m\frac{d^2y}{dt^2} = -ky$

$$m\frac{d^2y}{dt^2} = -ky$$

$$m\frac{d^2y}{dt^2} + ky = 0$$

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

$$\ddot{y} + \omega^2 y = 0 \qquad (5)$$

Here, $\omega^2 = \frac{k}{m}$. The characteristic equation for differential equation (5) has the following form:

$$k^2 + \omega^2 = 0 \qquad (6)$$

The solution of this quadratic equation is $k_{1,2} = \pm\omega \cdot i$. Thus, the general solution of equation (5) is:

$$y = C_1 e^{\omega t} \cdot \cos\omega t + C_2 e^{\omega t} \cdot \sin\omega t \tag{7}$$

Here, C_1 and C_2 are constant coefficients [2]. In the initial state of the body, its coordinate equals *A*, that is, when *t=0* da *y=A*. For this particular case:

$$A = C_1 e^{\omega \cdot 0} \cdot \cos\omega \cdot 0 + C_2 e^{\omega \cdot 0} \cdot \sin\omega \cdot 0$$

Thus, $C_1=A$. The body is initially at rest, meaning when *t=0, v=0*. The first derivative of the coordinate with respect to time is the velocity. Therefore, we differentiate equation (7) with respect to time and substitute t *t=0, v=0* to find C_2.

$$0 = -C_1\omega \cdot e^{\omega \cdot 0} \cdot \sin\omega \cdot 0 + C_2\omega \cdot e^{\omega \cdot 0} \cdot \cos\omega \cdot 0$$

From this equation, it follows that $C_2=0$. Substituting the obtainned values of C_1 and C_2 into equation (7), we get:

$$y = A \cdot \sin\omega t = 0{,}05\sin 10t \tag{8}$$

From equation (8), the first derivative gives the velocity and the second derivative gives the acceleration of the body:

$$\begin{cases} v = \dfrac{dy}{dt} = \omega A\cos\omega t = 0{,}5\cos 10t \\ a = \dfrac{d^2 y}{dt^2} = -\omega^2 A\sin\omega t = -5\sin 10t \end{cases} \tag{8$'$}$$

Problem 2. A car with a mass of *m=1,5* tons is moving with a uniform velocity of *v =20 m/s*. Suddenly, the engine stops working, and the car continues moving due to its inertia. If the air resistance coefficient acting on the car is *b = 3 kg/s*, determine its acceleration after *t = 5* seconds.

Solution: Let us take the point where the engine stops as the origin of coordinates (Fig.2). Since the driving force of the car becomes zero, it undergoes uniformly decelerated motion under the action of the resistance force *f=-bv*. According to Newton's second law, the net force acting on the car produces acceleration:

$$ma = -bv \tag{9}$$

Here, the negative sign indicates that the resistance force is directed opposite to the velocity of the car.

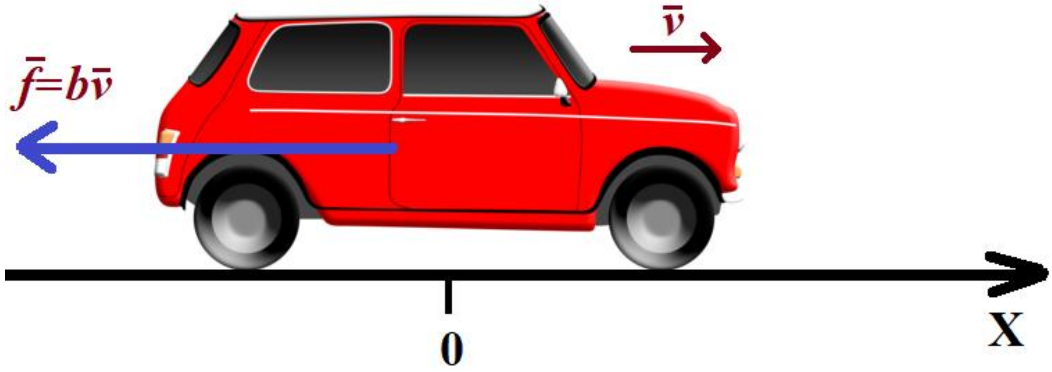

Fig2. A car moving under the effect of resistance force

The first-order derivative of the car's coordinate equation with respect to time gives its velocity, and the second-order derivative gives its acceleration, that is:

$$\begin{cases} v = \dfrac{dx}{dt} \\ a = \dfrac{d^2x}{dt^2} \end{cases} \qquad (10)$$

Using equations (9) and (10), we can formulate the following second-order differential equation for the motion of the vehicle:

$$ma + bv = 0$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} = 0$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\cdot\frac{dx}{dt} = 0$$

$$\ddot{x} + \frac{b}{m}\dot{x} = 0 \qquad (11)$$

The characteristic equation for this differential equation takes the form:

$$k^2 + \frac{b}{m}k = 0 \qquad (12)$$

The solution of this quadratic equation is $k_1=0,\ k_2=-\ b/m=-2\cdot10^{-3}\ 1/s$ Thus, the general solution of equation (12) can be expressed as:

$$x = C_1 e^{k_1 t} + C_2 e^{k_2 t} = C_1 + C_2 e^{-\frac{b}{m}t} \quad (13)$$

Here, C_1 and C_2 are constant coefficients, which are determined using the following conditions: at the start of observation (*t=0*), the vehicle is at the origin (*x=0*) and its velocity is at the maximum value (*v=20 m/s*).

$$\begin{cases} t = 0 \Rightarrow x = 0 \Rightarrow C_1 + C_2 \cdot e^{-2 \cdot 10^{-3} \cdot 0} = 0 \Rightarrow C_1 = -C_2 \\ t = 0 \Rightarrow v = 50 \Rightarrow v = \dfrac{dx}{dt} = -\dfrac{b}{m} \cdot C_2 \cdot e^{-\frac{b}{m} \cdot t} \Rightarrow 50 = -2 \cdot 10^{-3} \cdot C_2 \cdot e^{-2 \cdot 10^{-3} \cdot 0} \\ C_2 = -25 \cdot 10^3 m;\ C_1 = 25 \cdot 10^3 m \end{cases} \quad (14)$$

From equation (14), the time dependence of the vehicle's coordinate can be expressed as:

$$x = 25 \cdot 10^3 - 25 \cdot 10^3 e^{-2 \cdot 10^{-3} t} \quad (15)$$

To determine the acceleration of the body, we take the second derivative of the coordinate equation (15):

$$a = \frac{d^2 x}{dt^2} = C_2 \cdot \left(\frac{b}{m} \right)^2 \cdot e^{-\frac{b}{m}t} \Bigg|_{t=5} = -25 \cdot 10^3 \cdot 4 \cdot 10^{-6} \cdot e^{-2 \cdot 10^{-3} \cdot 5} = -0{,}1\ m/s^2 .$$

Problem 3. An electric circuit consists of the electromotive force *E=5 V*, the resistor with an active resistance of *R=20 Ω*, the capacitor with capacitance of *100 μF*, and the coil with inductance *of 25 mH* (Fig. 3). Find the equation for the current as a function of time after the switch is closed and determine the current at *t=0.25* seconds.

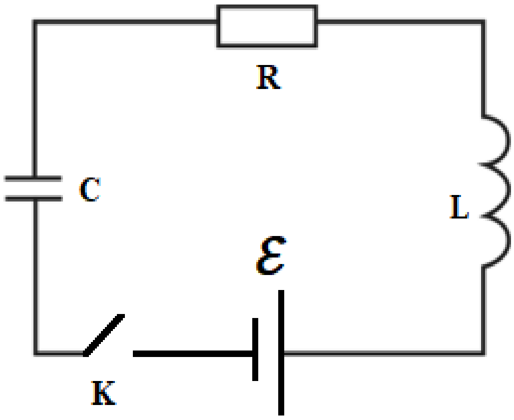

Fig3. A series-connected E, R, L, C electric circuit.

Solution: According to Kirchhoff's second law, the sum of the voltages across the resistor, capacitor, and inductor is equal to the electromotive force E:

$$E = U_R + U_C + U_L \qquad (16)$$

Here, $U_R = I \cdot R$, $U_C = \frac{q}{C} = \frac{I}{C}dt$, $U_L = L\frac{dI}{dt}$

Using these relationships, we can formulate the following equation:

$$E = IR + \frac{I}{C}dt + L\frac{dI}{dt} \qquad (17)$$

Taking the derivative of both sides of equation (17) with respect to time gives a second-order homogeneous differential equation for the current:

$$\frac{d^2 I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = 0 \qquad (18)$$

The characteristic equation of the differential equation (18) takes the form:

$$k^2 + \frac{R}{L}k + \frac{1}{LC} = 0 \qquad (19)$$

The roots $k_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -400 \pm 489i$ of this quadratic equation are complex numbers [3]. In this case, the general solution of equation (18) can be written as:

$$I = e^{-\frac{R}{2L}t}\left(C_1 \cos\omega t + C_2 \sin\omega t\right) \qquad (20)$$

Here $\omega = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$, C_1 and C_2 are constant coefficients to be determined using the initial conditions. At the moment the switch is closed ($t=0$), $I=0$ and $L\frac{dI}{dt}\Big|_{t=0} = E$

Using these conditions, the coefficients C_1 and C_2 can be found:

$$\begin{cases} t=0, I=0 \Rightarrow\ e^{-\frac{R}{2L}\cdot 0}\left(C_1\cos\omega\cdot 0+C_2\sin\omega\cdot 0\right)=0 \Rightarrow C_1=0 \\ t=0, \dfrac{dI}{dt}=0 \Rightarrow e^{-\frac{R}{2L}\cdot 0}\left[-\dfrac{R}{2L}\left(C_1\cos\omega t+C_2\sin\omega t\right)+\omega\left(-C_1\sin\omega t+C_2\cos\omega t\right)\right]=0 \Rightarrow C_2=\dfrac{E}{\omega L} \end{cases} \quad (21)$$

Using the determined coefficients C_1 and C_2 equation (20) can be written in the following form:

$$I=e^{-\frac{R}{2L}t}\cdot\frac{E}{\omega L}\sin\omega t \quad (22)$$

the value of the current at $t=0.25$ seconds is $I=e^{-400\cdot 0,25}\cdot\dfrac{5}{489\cdot 25\cdot 10^3}\sin 489\cdot 0,25\approx 5\,A$.

Conclusion. Final analyses show that modeling physical processes using second-order differential equations allows for a deeper understanding of their nature. The examples from mechanics, mechanical vibrations, and electrodynamics confirm that translating physical phenomena into mathematical form, analyzing them, and obtaining precise solutions is a convenient and effective method. This approach not only helps students reinforce theoretical concepts but also develops systematic and logical problem-solving skills in physics. Furthermore, it highlights the role and significance of differential equations in physics and demonstrates the efficiency of applying them in practice.

References

1. R.Turgunbayev, Sh. Ismailov. O. Abdullayev. Differensial tenglamalar kursidan misol va masalalar to'plami / Toshkent, TDPU, 2007 y.
2. Филиппов А. Ф. Сборник задач по дифференциальным уравнениям. Ижевск: НИЦ «Регулярная и хаотическая динамика», 2000
3. Гутер Р.С., Янпольский А.Р Дифференциальные уравнения. Высшая Школа. 1976.