CORRESPONDING TO SOME MODELS OF STATISTICAL MECHANICS P-ADIC DYNAMICS OF FUNCTIONS AND ITS APPLICATIONS:A NEW CONTRACTIVITY THEOREM FOR THE INHOMOGENEOUS POTTS-BETHE OPERATOR

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Abstract

In this paper, p–adic statistical mechanics of Potts-type models arising from nonlinear recurrent operators with inhomogeneous (spatially varying) parameters are studied. Unlike the homogeneous case on the Cayley tree widely covered in the literature, we consider the Potts–Bethe operator with interaction coefficients varying across each generation. As the main result, a new contractivity theorem is proved: if the appropriate Lipschitz bounds in the p-adic norm are satisfied, the operator maps the appropriate closed ball into itself and has a unique fixed point — this point corresponds to the unique‘‘p- adic Gibbs measure for the inhomogeneous model. The proof relies on ultrametric estimates, non-Archimedean analysis of multiplicative-divisive operators, and the‘‘‘p-adic analogue of the Banach fixed-point principle. As a result, a practical criterion for the absence of phase transitions is obtained, supported by numerical p-adic iteration algorithms.[1] Furthermore, a new research framework is proposed for poorly studied directions—inhomogeneous geometry, practical indicators, and applied modelling.

Keywords: p-adic dynamics, Potts model, Gibbs measure, Cayley tree, contractivity, inhomogeneous parameter, statistical mechanics‘.

Introduction. Over the past twenty years, *p*-adic analysis has become an important theoretical tool in statistical mechanics, dynamical systems, and hierarchical modelling. In particular, *p*-adic Gibbs o'lchovlar, faza o'tish, periodik va no-periodik yechimlar masalalari ko'plab ishlarda o'rganilgan. Klassik yo'nalishda model odatda bir jinsli parametrlarda (J=const) va Ceyley daraxt kabi aniq spatial non-uniformity of coefficients, local variation of the external field, and structural irregularities are natural in practical systems.[2-4]

A review of the above literature (*p*-adic Potts model, Potts-Bethe maps, quasi-Gibbs measures, chaos problems) reveals the following gap: a general and practical criterion combining contractivity and uniqueness in inhomogeneous-parameter Potts dynamics has not been sufficiently formalised. This paper is aimed precisely at filling this gap—proving the existence of a unique invariant point inside a p-adic ball for the inhomogeneous Potts-Bethe operator and linking this result to the uniqueness of the Gibbs measure.

the explicit functional form of the recurrent operator with inhomogeneous coefficients;

a new contractivity theorem (main theorem) and its complete proof are given;

a practical criterion for the absence of phase transitions is derived from the theorem;

p-adic iteration algorithm and a small numerical experiment table are presented.[4]

Problem Statement

*P*In p-adic statistical mechanics, the Gibbs measures, periodicity, and phase transition conditions of the Potts model have been studied in a number of works. However, most of the research has been:

1. restricted to homogeneous interactions ($J_n \equiv J$);
2. confined to the standard Cayley tree as the underlying geometry;
3. weakly connected existence/uniqueness results to numerical practical indicators.

In the present work we take the inhomogeneous case with { J_n } varying across the sequence. These parameters represent a physically heterogeneous medium across layers.

***p*-adic Basics:** $p-toq\, tub\, son$, Q_p p - adic field, $¿\cdot|_p$- p-adic norm. We work with the following unit ball:

$B_r=\{x\in Q_p:|x|_p\leq r\}$, $0<r<1.$

Ultrametric property:

$$|x+y|_p\leq max\{|x|_p,|y|_p\}.$$

Inhomogeneous Potts-Bethe operator: for the $q\geq 2$-state Potts model we denote the layer-wise message parameters by $h_n=\left(h_n^1,\ldots,h_n^{q-1}\right)\in Q_p^{q-1}$. For simplicity, in the scalar reduction we consider the following operator:[4-5]

$$\Phi_{n(x)}=\left(\frac{a_{nx}+b_n}{c_{nx}+d_n}\right)^k, \qquad (1)$$

where $k\in N$is the tree order, $a_n,b_b,c_n,d_n\in Q_p$ are inhomogeneous coefficients.

The full recurrent system:

$x_n=\Phi_n(x_{n+1}),n\geq 0$ (2)

To find the invariant regime, the corresponding composite operator:

$$T(x)=\left(\frac{A(x)}{C(x)}\right)^k, \qquad (3)$$

where A,C are the limit effective combinations of the inhomogeneous parameters. In practice we take $A(x)=¿$ $ax+b,C(x)=cx+d$ as the explicit form.

Defenation.1.$T:B_r\rightarrow B_r$ For an operator $0<\lambda<1$ there exists such that

$$|T(x)-T(y)|_p\leq\lambda|x-y|_p$$

For all $x,y\in B_r$ the operator is called p-adic contractiv.

To prove the theorem, the following steps are carried out:

1.$T(B_r)\subseteq B_r$ invariance estimate;

2. Ultrametric estimation of the Lipschitz bound for the Möbius-type term;

3. Uniqueness via the non-Archimedean version of the Banach fixed-point principle.

Auxiliary Lemmas

Lemma 1. (Ultrametric estimate for a quotient). If $u_i, v_i \in Q_p (i=1,2), |v_1|_p = |v_2|_p = 1$ $bo'lsa, unda$ [5]

$$\left|\frac{u_1}{v_1} - \frac{u_2}{v_2}\right|_p \le max\{|u_1 - u_2|_p, |v_1 - v_2|_p\}.$$

Proof.

$$\frac{u_1}{v_1} - \frac{u_2}{v_2} = \frac{u_1 v_2 - u_2 v_1}{v_1 v_2}.$$

$|v_1 v_2|_p$= 1, the denominator does not change the norm value. Writing the numerator as $u_1 v_2 \text{-} u_2 v_1 = (u_1 - u_2) v_2 + u_2 (v_2 - v_1)$ and applying ultrametricity yields the conclusion.

Lemma 2. (Estimate for a power function).[5-6] $Agar |x|_p \le r, |y|_p \le r \, va \, r < 1 \, bo' lsa,$

$$|x^k - y^k|_p \le |x - y|_p.$$

Proof.

$$x^k - y^k = (x - y) \sum_{j=0}^{k-1} x^{k-1-j} y^j.$$

Each term has norm $\le r^{k-1} < 1$,demak yig‘indining p-adik normasi ≤

Theorem 1. (Contractivity and uniqueness for the inhomogeneous Potts-Bethe operator). T Let the operator T be given by (3), where:

$$T(x) = \left(\frac{ax+b}{cx+d}\right)^k,$$

and let the following conditions hold*:*[6-7]

1. $|a|_p, |b|_p \le r, |c|_p \le r, |d|_p = 1, 0 < r < 1;$

2. $|ad - bc|_p \le r;$

3. $|cx + d|_p = 1 \, barcha \, x \in B_r uchun.$

Then:

1. $T(B_r) \subseteq B_r$;

2. $T B_r$ is contractive on B$_r$*;*

3. TT has a unique fixed point$x^{¿} \in B_r$;

4. the corresponding Potts recurrent equation (2) has a unique p-adic Gibbs state corresponding to the invariant solution.

Proof.

Step.1 **(invariance)**.$x \in B_r$. Let x∈B_r. Then $|ax+b|_p \le max\{|a|_p|x|_p, |b|_p\} \le r$. By assumption $|cx+d|_p=1$, hence

$$\left|\frac{ax+b}{cx+d}\right|_p \le r.$$

From this $|T(x)|_p = \left|\frac{ax+b}{cx+d}\right|_p^k \le r^k \le r$,i.e. $T(x) \in B_r$.

Step.2 **(Lipschitz estimate).** $F(x)=\frac{ax+b}{bx+d}$ Let us set F(x) = (ax+b)/(cx+d).

Then

$$F(x)-F(y)=\frac{(ad-bc)(x-y)}{(cx+d)(cy+d)}.$$

taking the norm:

$$|F(x)-F(y)|_p=|ad-bc|_p|x-y|_p \le r|x-y|_p,$$

since the denominator norm equals 1. Now from Lemma 2

$$T(x)-T(y)_p=\left|F(x)^k-F(y)^k\right|_p \le |F(x)-F(y)|_p \le r|x-y|_p.$$

we obtain the above, so the contractivity coefficient is: $\lambda \le r < 1$.which gives the result.

Step.3 (**unique fixed point**). $B_r$$B_r$ is a complete space with respect to the p-adic norm.

By the fixed-point theorem, T has a fixed point $x^{¿}$. For any $x_0 \in B_r$ $x_{n+1}=T(x_n)$ the sequence $x^{¿}$ converges to it.

Step.4 **(Gibbs interpretation).** The recurrent equations of the Potts model in the p-adic setting parametrise Gibbs measures via boundary laws. Uniqueness of the fixed point means uniqueness of this boundary law, which in turn implies uniqueness of the Gibbs state. This completes the proof.

Corollary.1. (Criterion for the absence of a phase transition). When the conditions of Theorem 1 hold, multiple Gibbs states do not appear in the translation-invariant class, i.e. no phase transition is observed.

Proof. The standard criterion for a phase transition is the existence of at least two extremal Gibbs states. Theorem 1 yields a unique Gibbs state, so no such splitting occurs.[7]

The simple iteration arising from the theorem:

$$x^{m+1} = T\left(x^{(m)}\right), m = 0,1,2,\ldots.$$

The main theorem relaxes several constraints from previous works.

Covering inhomogeneity. While in the classical homogeneous Potts case parameters are constant, this work is especially important for systems with inhomogeneous coefficients and a contractivity criterion.

Linking dynamics and Gibbs measure. In many works the phase transition problem and the properties of the dynamic map are treated separately. In our approach, uniqueness of the fixed point is transferred directly to uniqueness of the Gibbs state.

Practical application. The proposed criterion allows pre-filtering of model parameters: if the contractivity conditions are satisfied, multiple states are not expected in the model. This reduces numerical computation costs.

Limitations. The paper mainly considers the one-dimensional reduction (scalar message). For the full $(q-1)$-dimensional vector state, stronger techniques involving Jacobian estimates and p-adic matrix norms are required. Moreover, the systematic classification of cases where phase transitions do occur remains a separate problem.

Conclusion. A new contractivity theorem for the inhomogeneous p-adic Potts-type model of statistical mechanics was proposed and fully proved. According to the theorem, the invariance, contractivity, and unique fixed point of the operator inside the appropriate p-adic ball are guaranteed. This result establishes the existence of a unique Gibbs state of the model and serves as a precise criterion for the absence of a phase transition.

In this way, the direction poorly covered in previous works on the topic—a unique and practically applicable dynamic criterion for inhomogeneous parameters—was advanced from both theoretical and computational perspectivesblash nuqtai nazaridan rivojlantirildi.

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