**ILL-POSED INITIAL–BOUNDARY VALUE PROBLEM FOR A SYSTEM OF TIME-DIRECTION CHANGING PARABOLIC-TYPE EQUATIONS**

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**Abstract.** This paper is devoted to the investigation of an ill-posed initial–boundary value problem, in the sense of Hadamard, for a system of time-direction changing parabolic-type equations. By applying the Fourier method and the spectral properties of the corresponding differential operator, an appropriate correctness set is constructed for the considered problem. Sufficient conditions guaranteeing the uniqueness of the solution in this set are established. In addition, conditional stability estimates are derived by means of suitable a priori inequalities. The obtained results extend the theory of ill-posed problems for systems of time-direction changing parabolic-type equations and may serve as a theoretical foundation for further studies of inverse and nonlocal boundary value problems.

**Keywords.** system of time-direction changing parabolic-type equations, ill-posed initial–boundary value problem, correctness set, uniqueness of solution, conditional stability, a priori estimates, Fourier method, spectral problem.

**НЕКОРРЕКТНАЯ НАЧАЛЬНО-КРАЕВАЯ ЗАДАЧА ДЛЯ СИСТЕМЫ ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ С ИЗМЕНЯЮЩИМСЯ ПО ВРЕМЕНИ НАПРАВЛЕНИЕМ**

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**Аннотация.** Настоящая работа посвящена исследованию некорректной начально-краевой задачи в смысле Ж. Адамара для системы параболических уравнений с изменяющимся по времени направлением. С использованием метода Фурье и спектральных свойств соответствующего дифференциального оператора для рассматриваемой задачи построено соответствующее множество корректности. Установлены достаточные условия, гарантирующие единственность решения в этом множестве. Кроме того, посредством соответствующих априорных неравенств получены условные оценки устойчивости решения. Полученные результаты развивают теорию некорректных задач для систем параболических уравнений с изменяющимся по времени направлением и могут служить теоретической основой для дальнейших исследований обратных и нелокальных краевых задач.

**Ключевые слова.** Система параболических уравнений с изменяющимся по времени направлением, некорректная начально-краевая задача, множество корректности, единственность решения, условная устойчивость, априорные оценки, метод Фурье, спектральная задача.

## Introduction

Time-direction changing parabolic-type differential equations and their systems play an important role in modeling various processes arising in mathematical physics, heat conduction theory, diffusion phenomena, filtration processes, control theory, and many other fields of applied science. A distinctive feature of such equations is that their evolution direction changes over different time intervals. Consequently, the mathematical properties of the corresponding problems become considerably more complicated, making the investigation of the existence, uniqueness, and stability of solutions a problem of significant theoretical importance.

According to Hadamard's definition, a problem is said to be ill-posed if at least one of the following conditions is violated: the existence of a solution, its uniqueness, or its continuous dependence on the given data. The theory of ill-posed problems was initiated by J. Hadamard and was subsequently developed by A.N. Tikhonov[1], M.M. Lavrentiev[2], V.K. Ivanov[3], A.I. Kozhanov[4], S.G.

Pyatkov[5,6], S.A. Tersenov[7], K.S. Fayazov[8], A.A. Kerefov[9], I.E. Egorov[10], I.O. Khajiyev[11], Y.K. Khudayberganov[12,13] and many researches, furtheremore this problem's non-linear types are discussed in these [17,18] works. Their works have established the theoretical foundations for the study of correctness classes, uniqueness, and conditional stability of solutions to ill-posed initial and boundary value problems for various classes of differential equations.

In recent years, increasing attention has been devoted to the investigation of initial-boundary value problems for time-direction changing parabolic-type equations. Such equations describe processes that evolve forward in time over certain intervals and backward in time over others. Consequently, the analysis of their solutions by classical methods is often inadequate and requires the application of special functional spaces, spectral methods, and appropriate a priori estimates. In particular, ill-posed initial-boundary value problems for systems of time-direction changing parabolic-type equations have not yet been sufficiently investigated, which makes this research area both challenging and relevant.

In the present paper, an ill-posed initial-boundary value problem, in the sense of Hadamard, is investigated for a system of time-direction changing parabolic-type equations. An appropriate correctness set is introduced for the considered problem, and sufficient conditions ensuring the uniqueness of the solution within this set are established. Furthermore, conditional stability estimates are derived by means of suitable a priori estimates. The obtained results extend the theory of ill-posed problems for systems of time-direction changing parabolic-type equations and provide a theoretical foundation for further investigations of this class of initial-boundary value problems.

**Problem Statement.**

Suppose that a pair of functions $(u_1(x,y,t), u_2(x,y,t))$ is a solution to the system of partial differential equations of parabolic type in the domain $\Omega = \Omega_0 \times (0;T)$:

$$\begin{cases} u_{1t}(x,y,t)+sgn(x)u_{1xx}(x,y,t)+sgn(y)u_{1yy}(x,y,t)+au_2(x,y,t)=0, \\ u_{2t}(x,y,t)+sgn(x)u_{2xx}(x,y,t)+sgn(y)u_{2yy}(x,y,t)+bu_1(x,y,t)=0, \end{cases} \quad (1)$$

where $a,b$ are arbitrary constants, $a\cdot b>0$ and $\Omega_0=\{x,y\,|\,(-1;1)^2, x\neq 0, y\neq 0\}$, $T<\infty$.

The system of equations (1) in the domain $\Omega$ and the following:

$$u_j(x,y,t)\big|_{t=0}=\varphi_i(x,y),(x,y)\in[-1;1]^2, \quad (2)$$

initial,

$$u_j(x,y,t)\Big|_{\substack{x=-1\\x=+1}}=0,\ (y,t)\in[-1;1]\times\bar{Q},$$

$$u_j(x,y,t)\Big|_{\substack{y=-1\\y=+1}}=0,\ (x,t)\in[-1;1]\times\bar{Q}, \quad (3)$$

boundary,

$$\left.\frac{\partial^i u_j(x,y,t)}{\partial x^i}\right|_{x=-0}=\left.\frac{\partial^i u_j(x,y,t)}{\partial x^i}\right|_{x=+0},(y,t)\in[-1;1]\times\bar{Q},$$

$$\left.\frac{\partial^i u_j(x,y,t)}{\partial y^i}\right|_{y=-0}=\left.\frac{\partial^i u_j(x,y,t)}{\partial y^i}\right|_{y=+0},(x,t)\in[-1;1]\times\bar{Q}, \quad (4)$$

and satisfying the gluing conditions, find a pair of continuous functions $(u_1(x,y,t),\,u_2(x,y,t))$, where $\left(i=\overline{0,1},\ j=\overline{1,2}\right)$ and $\varphi_1(x,y),\ \varphi_2(x,y)$ are given sufficiently smooth functions, furthermore, $\varphi_i(x,y)\big|_{\partial\Omega_0}=0.$

To solve the problem (1)–(4), we perform the following:

$$u_1(x,y,t)=\frac{1}{2b}e^{-\sqrt{ab}t}\omega(x,y,t)+\frac{1}{2b}e^{\sqrt{ab}t}\varpi(x,y,t)$$

$$u_2(x,y,t)=\frac{1}{2\sqrt{ab}}e^{-\sqrt{ab}t}\omega(x,y,t)-\frac{1}{2\sqrt{ab}}e^{\sqrt{ab}t}\varpi(x,y,t) \quad (5)$$

substitution. As a result, in finding the solution to the problem under consideration, with respect to the functions $\omega(x,y,t)$ and $\varpi(x,y,t)$ we have the following:

**Problem 1.** The following

$$\omega_t(x,y,t)+sgn(x)\omega_{xx}(x,y,t)+sgn(y)\omega_{yy}(x,y,t)=0 \tag{6}$$

equation in the domain $\Omega$ as follows:

initial

$$\omega(x,y,t)\big|_{t=0}=\overline{\varphi}_0(x,y),(x,y)\in[-1;1]^2 \tag{7}$$

boundary

$$\omega(x,y,t)\Big|_{\substack{x=-1\\x=+1}}=0,\ (y,t)\in[-1;1]\times\overline{Q},$$

$$\omega(x,y,t)\Big|_{\substack{y=-1\\y=+1}}=0,\ (x,t)\in[-1;1]\times\overline{Q}, \tag{8}$$

and gluing

$$\left.\frac{\partial^i\omega(x,y,t)}{\partial x^i}\right|_{x=-0}=\left.\frac{\partial^i\omega(x,y,t)}{\partial x^i}\right|_{x=+0},(y,t)\in[-1;1]\times\overline{Q},$$

$$\left.\frac{\partial^i\omega(x,y,t)}{\partial y^i}\right|_{y=-0}=\left.\frac{\partial^i\omega(x,y,t)}{\partial y^i}\right|_{y=+0},(x,t)\in[-1;1]\times\overline{Q},i=0,1 \tag{9}$$

conditions, where $\overline{\varphi_0}(x,y)=b\varphi_0(x,y)+\sqrt{ab}\varphi_1(x,y)$.

**Problem 2.** The following

$$\varpi_t(x,y,t)+sgn(x)\varpi_{xx}(x,y,t)+sgn(y)\varpi_{yy}(x,y,t)=0 \tag{10}$$

equation in the domain $\Omega$ as follows:

$$\varpi(x,y,t)\big|_{t=0}=\overline{\varphi}_1(x,y),(x,y)\in[-1;1]^2, \tag{11}$$

initial,

$$\varpi(x,y,t)\Big|_{\substack{x=-1\\x=+1}}=0,\ (y,t)\in[-1;1]\times\overline{Q},$$

$$\varpi(x,y,t)\Big|_{\substack{y=-1\\y=+1}}=0,\ (x,t)\in[-1;1]\times\overline{Q}, \tag{12}$$

boundary,

$$\left.\frac{\partial^i \varpi(x,y,t)}{\partial x^i}\right|_{x=-0} = \left.\frac{\partial^i \varpi(x,y,t)}{\partial x^i}\right|_{x=+0}, (y,t)\in[-1;1]\times\bar{Q},$$
$$\left.\frac{\partial^i \varpi(x,y,t)}{\partial y^i}\right|_{y=-0} = \left.\frac{\partial^i \varpi(x,y,t)}{\partial y^i}\right|_{y=+0}, (x,t)\in[-1;1]\times\bar{Q}, i=0,1 \tag{13}$$

and satisfying the gluing conditions, leads to the problem of finding the functions $\omega(x,y,t)$ and $\varpi(x,y,t)$, where $\overline{\varphi_1}(x,y)=b\varphi_0(x,y)-\sqrt{ab}\varphi_1(x,y)$.

**Spectral problem**

Find the values of the parameter $\lambda$ for which the following:

$$sgn(x)\vartheta_{xx}(x,y)+sgn(y)\vartheta_{yy}(x,y)+\lambda\vartheta(x,y)=0,\ (x,y)\in\Omega_0, \tag{14}$$

$$\vartheta(x,y)\Big|_{\substack{x=-1\\x=+1}}=0,\ y\in[-1;1],\quad \vartheta(x,y)\Big|_{\substack{y=-1\\y=+1}}=0,\quad x\in[-1;1],$$
$$\left.\frac{\partial^i \vartheta(x,y)}{\partial x^i}\right|_{x=-0} = \left.\frac{\partial^i \vartheta(x,y)}{\partial x^i}\right|_{x=+0},\ y\in[-1;1],$$
$$\left.\frac{\partial^i \vartheta(x,y)}{\partial y^i}\right|_{y=-0} = \left.\frac{\partial^i \vartheta(x,y)}{\partial y^i}\right|_{y=+0},\ x\in[-1;1],\ \left(i=\overline{0,1}\right), \tag{15}$$

problem has a non-trivial solution.

We use the Fourier method to solve the problem (14), (15). That is,

$$\vartheta(x,y)=X(x)\cdot Y(y). \tag{16}$$

Using (16), we obtain the following conditions:

$$X(-1)=X(+1)=0,$$
$$X(-0)=X(+0),$$
$$X'(-0)=X'(+0), \tag{17}$$

$$Y(-1)=Y(+1)=0,$$
$$Y(-0)=Y(+0),$$
$$Y'(-0)=Y'(+0). \tag{18}$$

Calculating the second-order derivatives of (16):

$$\vartheta_{xx}(x,y)=X''(x)\cdot Y(y),$$
$$\vartheta_{yy}(x,y)=X(x)\cdot Y''(y),\ \vartheta_{yy}(x,y)=X(x)\cdot Y(y).$$

Substituting into equation (14), we obtain the following:

$$\frac{sgn(x)X''(x)}{X(x)}+\frac{sgn(y)Y''(y)}{Y(y)}=-\lambda.$$

equality. Since the expression $\frac{sgn(x)X''(x)}{X(x)}$ in the equality above does not depend on $y$, and the expression $\frac{sgn(y)Y''(y)}{Y(y)}$ does not depend on $x$, we have the following:

$$\frac{sgn(x)X''(x)}{X(x)}=-\lambda_1, \quad \frac{sgn(y)Y''(y)}{Y(y)}=-\lambda_2$$

equalities.

From this, for $X(x)$ and $Y(y)$ we have the following equations:

$$sgn(x)X''(x)=-\lambda_1 X(x), \tag{19}$$

$$sgn(y)Y''(y)=-\lambda_2 Y(y). \tag{20}$$

We consider equations (19) and (20) with conditions (17) and (18), respectively:

$$\begin{aligned} &sgn(x)X''(x)=-\lambda_1 X(x),\\ &X(-1)=X(+1)=0,\\ &X(-0)=X(+0),\\ &X'(-0)=X'(+0),\end{aligned} \tag{21}$$

$$\begin{aligned} &sgn(y)Y''(y)=-\lambda_2 Y(y),\\ &Y(-1)=Y(+1)=0,\\ &Y(-0)=Y(+0),\\ &Y'(-0)=Y'(+0).\end{aligned} \tag{22}$$

The solutions to the problems (21) and (22)

if $\lambda_1>0,\ \lambda_2>0,$

$$X_k^{(1)}(x)=\begin{cases}\sin\mu_k(x-1)/\cos\mu_k \quad ,0\le x\le 1,\\ sh\mu_k(x+1)/ch\mu_k,-1\le x\le 0,\end{cases} k\in N,$$

$$Y_l^{(1)}(y)=\begin{cases}\sin\sigma_l(y-1)/\cos\sigma_l,0\le y\le 1,\\ sh\sigma_l(y+1)/ch\sigma_l,-1\le y\le 0,\end{cases} l\in N,$$

if $\lambda_1<0,\lambda_2<0,$

$$X_k^{(2)}(x)=\begin{cases}sh\mu_k(x-1)/ch\mu_k,0\le x\le 1,\\ \sin\mu_k(x+1)/\cos\mu_k,-1\le x\le 0,\end{cases} k\in N,$$

$$Y_l^{(2)}(y)=\begin{cases}sh\sigma_l(y-1)/ch\sigma_l \ ,0\le y\le 1,\\ \sin\sigma_l(y+1)/\cos\sigma_l \ ,-1\le y\le 0,\end{cases} l\in N.$$

where $\lambda_{1k}=\mu_k^2>0,\ \lambda_{1k}=-\mu_k^2<0,\ \ \lambda_{2l}=\sigma_l^2>0,\ \lambda_{2l}=-\sigma_l^2<0$ are in the following form.

In both cases, $\mu_k, \sigma_l$ are the positive roots of the following $tg\alpha=-th\alpha$ the transcendental equation.

Thus, the eigenvalues of the spectral problem (14), (15) are $\mu_k^2+\sigma_l^2=\lambda_{k,l}^{(1)},\ \mu_k^2-\sigma_l^2=\lambda_{k,l}^{(2)},\ -\mu_k^2+\sigma_l^2=\lambda_{k,l}^{(3)},\ -\mu_k^2-\sigma_l^2=\lambda_{k,l}^{(4)}$

in the form, and the corresponding eigenfunctions are

$$\vartheta_{k,l}^{(1)}(x,y)=X_k^{(1)}(x)\cdot Y_l^{(1)}(y),$$
$$\vartheta_{k,l}^{(2)}(x,y)=X_k^{(1)}(x)\cdot Y_l^{(2)}(y),$$
$$\vartheta_{k,l}^{(3)}(x,y)=X_k^{(2)}(x)\cdot Y_l^{(1)}(y),$$
$$\vartheta_{k,l}^{(4)}(x,y)=X_k^{(2)}(x)\cdot Y_l^{(2)}(y),\ \ k,l\in N,$$

in the form.

According to the result of this [14] work, the eigenvalues of the spectral problem (14), (15) $\left\{\lambda_{k,l}^{(1)}\right\}_{k,l=1}^{\infty},\left\{-\lambda_{k,l}^{(2)}\right\}_{k,l=1}^{\infty},\left\{\lambda_{k,l}^{(3)}\right\}_{k,l=1}^{\infty},\left\{-\lambda_{k,l}^{(4)}\right\}_{k,l=1}^{\infty}$ form an increasing sequence and the eigenfunctions are $\left\{\vartheta_{k,l}^{(j)}(x,y)\right\}_{k,l=1}^{\infty},\left(j=\overline{1,4}\right)$.

Let $\|u\|^2=(u,u)$, where $(u,\upsilon)=\int\limits_{-1}^{1}\int\limits_{-1}^{1}u\upsilon dxdy,$ furthermore,

$$\left(sgn(x)sgn(y)\vartheta_{k,l}^{(p)}(x,y),\vartheta_{i,j}^{(q)}(x,y)\right)=0,\ p\neq q,\ (p,q=\overline{1,4}),\ \forall\, k,l,i,j,$$

$$\left(sgn(x)sgn(y)\vartheta_{k,l}^{(m)}(x,y),\vartheta_{i,j}^{(m)}(x,y)\right)=\begin{cases}1,\ k=i\wedge l=j,\\0,\ k\neq i\wedge l\neq j,\end{cases}\ (m=1,4),$$

$$\left(sgn(x)sgn(y)\vartheta_{k,l}^{(m)}(x,y),\vartheta_{i,j}^{(m)}(x,y)\right)=\begin{cases}-1,\ k=i\wedge l=j\\0,\ k\neq i\wedge l\neq j\end{cases},(m=2,3),$$

where $k,l,i,j\in N.$

The norm defined by the following formula:

$$\|u(x,y,t)\|_0^2=\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left\{\left|\left(sgn(x)sgn(y)u(x,y,t),\vartheta_{k,l}^{(1)}(x,y)\right)\right|^2+\right.$$
$$+\left|\left(sgn(x)sgn(y)u(x,y,t),\vartheta_{k,l}^{(2)}(x,y)\right)\right|^2+$$
$$+\left|\left(sgn(x)sgn(y)u(x,y,t),\vartheta_{k,l}^{(3)}(x,y)\right)\right|^2+$$
$$\left.+\left|\left(sgn(x)sgn(y)u(x,y,t),\vartheta_{k,l}^{(4)}(x,y)\right)\right|^2\right\}, \tag{23}$$

is equivalent to the initial norm in the space $H_0$.

According to the results of theorem 2.1 in [16] and theorem 4.1 in [15], the eigenfunctions $\left\{\vartheta_{k,l}^{(j)}(x,y)\right\},\ \left(j=\overline{1,4}\right)$ of the spectral problem (14), (15) constitute a normalized Riesz basis in $L_2\left((-1;1)\times(-1;1)\right)$

**A priori estimate**

**Lemma 1.** Suppose that the function $\omega(x,y,t)$ satisfies the equation (6) and the conditions (7)–(9) in the domain $\Omega$. Then, for the function $\omega(x,y,t)$, the following inequality holds for $t\in(0;T)$:

$$\|\omega(x,y,t)\|_0\le 2\|\omega(x,y,0)\|_0^{\frac{T-t}{T}}\cdot\|\omega(x,y,T)\|_0^{\frac{t}{T}}$$

The proof of Lemma 1 can be found in [8].

**Theorem 1.** Suppose that $(u_1(x,y,t),\,u_2(x,y,t))$ is a solution to the problem (1)–(4). Then, for the functions $u_1(x,y,t),\,u_2(x,y,t)$, the following inequalities hold for $t\in(0;\ T)$:

$$\|u_1(x,y,t)\|_0\le\frac{1}{|b|}\left(\left(\left(\|\overline{\varphi}_0(x,y)\|_0\right)^{1-\frac{t}{T}}+\left(\|\overline{\varphi}_1(x,y)\|_0\right)^{1-\frac{t}{T}}\right)\left(|b|\|u_1(x,y,T)\|_0+\sqrt{ab}\,\|u_2(x,y,T)\|_0\right)^{\frac{t}{T}}\right),$$

$$\|u_2(x,y,t)\|_0\le\frac{1}{\sqrt{ab}}\left(\left(\left(\|\overline{\varphi}_0(x,y)\|_0\right)^{1-\frac{t}{T}}+\left(\|\overline{\varphi}_1(x,y)\|_0\right)^{1-\frac{t}{T}}\right)\left(|b|\|u_1(x,y,T)\|_0+\sqrt{ab}\,\|u_2(x,y,T)\|_0\right)^{\frac{t}{T}}\right)$$

**Proof.** By Lemma 1, the following inequalities hold for the functions $\omega(x,y,t)$ and $\varpi(x,y,t)$:

$$\|\omega(x,y,t)\|_0\le 2\|\omega(x,y,0)\|_0^{\frac{T-t}{T}}\cdot\|\omega(x,y,T)\|_0^{\frac{t}{T}}, \tag{24}$$

$$\left\|\varpi(x,y,t)\right\|_0 \le 2\left\|\varpi(x,y,0)\right\|_0^{\frac{T-t}{T}} \cdot \left\|\varpi(x,y,T)\right\|_0^{\frac{t}{T}}. \tag{25}$$

Using (5), we obtain the following equalities:

$$\omega(x,y,t) = \left(bu_1(x,y,t) + \sqrt{ab}u_2(x,y,t)\right)e^{\sqrt{ab}t},$$
$$\varpi(x,y,t) = \left(bu_1(x,y,t) - \sqrt{ab}u_2(x,y,t)\right)e^{-\sqrt{ab}t}. \tag{26}$$

Using the inequalities (24), (25) and the equalities (26), we obtain the following inequalities:

$$\left\|\omega(x,y,t)\right\|_0 \le 2\left\|\overline{\varphi}_0(x,y)\right\|_0^{1-\frac{t}{T}}\left\|bu_1(x,y,T) + \sqrt{ab}u_2(x,y,T)\right\|_0^{\frac{t}{T}} e^{\sqrt{ab}t}, \tag{27}$$

$$\left\|\varpi(x,y,t)\right\|_0 \le 2\left\|\overline{\varphi}_1(x,y)\right\|_0^{1-\frac{t}{T}}\left\|bu_1(x,y,T) - \sqrt{ab}u_2(x,y,T)\right\|_0^{\frac{t}{T}} e^{-\sqrt{ab}t}. \tag{28}$$

Based on the equalities (5), the following estimates for the norms of the functions $u_1(x,y,t)$ and $u_2(x,y,t)$ hold:

$$\left\|u_1(x,y,t)\right\|_0 \le \frac{1}{2|b|}\left(e^{-\sqrt{ab}t}\left\|\omega(x,y,t)\right\|_0 + e^{\sqrt{ab}t}\left\|\varpi(x,y,t)\right\|_0\right),$$
$$\left\|u_2(x,y,t)\right\|_0 \le \frac{1}{2\sqrt{ab}}\left(e^{-\sqrt{ab}t}\left\|\omega(x,y,t)\right\|_0 + e^{\sqrt{ab}t}\left\|\varpi(x,y,t)\right\|_0\right). \tag{29}$$

By using the inequalities (27), (28), and (29), we arrive at the following estimates:

$$\left\|u_1(x,y,t)\right\|_0 \le \frac{1}{|b|}\left(\left(\left\|\overline{\varphi}_0(x,y)\right\|_0\right)^{1-\frac{t}{T}} + \left(\left\|\overline{\varphi}_1(x,y)\right\|_0\right)^{1-\frac{t}{T}}\right)\left(|b|\left\|u_1(x,y,T)\right\|_0 + \sqrt{ab}\left\|u_2(x,y,T)\right\|_0\right)^{\frac{t}{T}},$$
$$\left\|u_2(x,y,t)\right\|_0 \le \frac{1}{\sqrt{ab}}\left(\left(\left\|\overline{\varphi}_0(x,y)\right\|_0\right)^{1-\frac{t}{T}} + \left(\left\|\overline{\varphi}_1(x,y)\right\|_0\right)^{1-\frac{t}{T}}\right)\left(|b|\left\|u_1(x,y,T)\right\|_0 + \sqrt{ab}\left\|u_2(x,y,T)\right\|_0\right)^{\frac{t}{T}}. \tag{30}$$

This concludes the proof of Theorem 1.

Consequently, we define the set of well-posedness for the problem (1)–(4) as:

$$M = \left\{\left(u_1(x,y,t), u_2(x,y,t)\right) : \left\|u_1(x,y,T)\right\|_0 + \left\|u_2(x,y,T)\right\|_0 < m, m < \infty\right\}, \tag{31}$$

**Uniqueness and Conditional Stability**

**Theorem 2.** If a solution to the problem (1)–(4) exists and $\left(u_1(x,y,t), u_2(x,y,t)\right) \in M$, then the solution to the problem (1)–(4) is unique.

**Proof.** Assume that the pair of functions $\left(u_{1,1}(x,y,t),\, u_{2,1}(x,y,t)\right)$, $\left(u_{1,2}(x,y,t), u_{2,2}(x,y,t)\right)$, are solutions to the problem (1)–(4). We introduce the notation $U_1(x,y,t)=u_{1,1}(x,y,t)-u_{1,2}(x,y,t)$ and $U_2(x,y,t)=u_{2,1}(x,y,t)-u_{2,2}(x,y,t)$. Then, the pair of functions $\left(U_1(x,y,t),\, U_2(x,y,t)\right)$ satisfies the system of equations:

$$\begin{cases} U_{1t}+sgn(x)U_{1xx}+sgn(y)U_{1yy}+aU_2=0, \\ U_{2t}+sgn(x)U_{2xx}+sgn(y)U_{2yy}+bU_1=0, \end{cases} \tag{32}$$

and the initial conditions:

$$U_j(x,y,t)\big|_{t=0}=0, (x,y)\in[-1;1]^2, \tag{33}$$

as well as the boundary conditions:

$$U_j(x,y,t)\Big|_{\substack{x=-1\\x=+1}}=0,\ (y,t)\in[-1;1]\times\bar{Q},$$

$$U_j(x,y,t)\Big|_{\substack{y=-1\\y=+1}}=0,\ (x,t)\in[-1;1]\times\bar{Q}, \tag{34}$$

By applying the inequalities (30) to the solution of the problem (32)–(34), it follows that:

$$\left\|U_1(x,y,t)\right\|=0,\ \left\|U_2(x,y,t)\right\|=0$$

Therefore, for any $(x,y,t)\in\Omega$ we have

$$u_{1,1}(x,y,t)\ \equiv\ u_{1,2}(x,y,t), u_{2,1}(x,y,t)\ \equiv u_{2,2}(x,y,t).$$

This completes the proof of Theorem 2.

Let $\left(u_1(x,y,t),\, u_2(x,y,t)\right)$ be the pair of functions representing the solution to the problem (1)–(4) with exact data, and let $\left(u_{1\varepsilon}(x,y,t), u_{2\varepsilon}(x,y,t)\right)$ be the pair of functions representing the solution to the problem (1)–(4) with approximate data.

**Theorem 3.** Suppose that $\left(u_1(x,y,t),u_2(x,y,t)\right)\in M,\ \left(u_{1\varepsilon}(x,y,t),u_{2\varepsilon}(x,y,t)\right)\in M$ and $\left\|\varphi_i(x,y)-\varphi_{i\varepsilon}(x,y)\right\|_0\le\varepsilon,\ i=0,1.$ Then, for the functions $U_{1\varepsilon}(x,y,t),\ U_{2\varepsilon}(x,y,t)$ the following estimates hold for $t\in(0;T)$:

$$\left\|U_{1\varepsilon}(x,y,t)\right\|_0\le\frac{2}{|b|}\left(\left(\varepsilon\left(|b|+\sqrt{ab}\right)\right)^{1-\frac{t}{T}}\left(2m\left(|b|+\sqrt{ab}\right)\right)^{\frac{t}{T}}\right),$$

$$\left\|U_{2\varepsilon}(x,y,t)\right\|_0\le\frac{2}{\sqrt{ab}}\left(\left(\varepsilon\left(|b|+\sqrt{ab}\right)\right)^{1-\frac{t}{T}}\left(2m\left(|b|+\sqrt{ab}\right)\right)^{\frac{t}{T}}\right).$$

**Proof.** We introduce the notation $U_{1\varepsilon}(x,y,t)=u_1(x,y,t)-u_{1\varepsilon}(x,y,t)$, $U_{2\varepsilon}(x,y,t)=u_2(x,y,t)-u_{2\varepsilon}(x,y,t)$. Then, for the pair of functions $(U_{1\varepsilon}(x,y,t),U_{2\varepsilon}(x,y,t))$ we have the following system of equations:

$$\begin{cases}U_{1\varepsilon t}+sgn(x)U_{1\varepsilon xx}+sgn(y)U_{1\varepsilon yy}+aU_{2\varepsilon}=0,\\ U_{2\varepsilon t}+sgn(x)U_{2\varepsilon xx}+sgn(y)U_{2\varepsilon yy}+bU_{1\varepsilon}=0,\end{cases}\tag{35}$$

and the initial conditions take the form:

$$\begin{aligned}&U_{1\varepsilon}(x,y,0)=\varphi_0(x,y)-\varphi_{0\varepsilon}(x,y)\\ &U_{2\varepsilon}(x,y,0)=\varphi_1(x,y)-\varphi_{1\varepsilon}(x,y).\end{aligned}\tag{36}$$

Taking into account that $\left(u_1(x,y,t),u_2(x,y,t)\right)\in M,\ \left(u_{1\varepsilon}(x,y,t),u_{2\varepsilon}(x,y,t)\right)\in M$ in the problem (35), (36), we perform the following estimations:

$$\left\|\overline{\varphi_{0\varepsilon}}(x,y)\right\|_0=|b|\left\|\varphi_0(x,y)-\varphi_{0\varepsilon}(x,y)\right\|_0+\sqrt{ab}\left\|\varphi_1(x,y)-\varphi_{1\varepsilon}(x,y)\right\|_0\le\varepsilon\left(|b|+\sqrt{ab}\right)$$

$$\left\|\overline{\varphi_{1\varepsilon}}(x,y,z)\right\|_0=|b|\left\|\varphi_0(x,y)-\varphi_{0\varepsilon}(x,y)\right\|_0-\sqrt{ab}\left\|\varphi_1(x,y)-\varphi_{1\varepsilon}(x,y)\right\|_0\le\varepsilon\left(|b|-\sqrt{ab}\right)\le\varepsilon\left(|b|+\sqrt{ab}\right).$$

$$\begin{aligned}&|b|\left\|U_{1\varepsilon}(x,y,T)\right\|_0+\sqrt{ab}\left\|U_{2\varepsilon}(x,y,T)\right\|_0=|b|\left\|u_1(x,y,T)-u_{1\varepsilon}(x,y,T)\right\|_0+\\ &+\sqrt{ab}\left\|u_2(x,y,T)-u_{2\varepsilon}(x,y,T)\right\|_0\le|b|\left(\left\|u_1(x,y,T)\right\|_0+\left\|u_{1\varepsilon}(x,y,T)\right\|_0\right)+\\ &+\sqrt{ab}\left(\left\|u_2(x,y,T)\right\|_0+\left\|u_{2\varepsilon}(x,y,T)\right\|_0\right)\le2m\left(|b|+\sqrt{ab}\right),\end{aligned}$$

and,

$$|b|\left\|U_{1\varepsilon}(x,y,T)\right\|_0-\sqrt{ab}\left\|U_{2\varepsilon}(x,y,T)\right\|_0=|b|\left\|u_1(x,y,T)-u_{1\varepsilon}(x,y,T)\right\|_0-$$

$$\sqrt{ab}\left\|u_2(x,y,T)-u_{2\varepsilon}(x,y,T)\right\|_0\le|b|\left(\left\|u_1(x,y,T)\right\|_0+\left\|u_{1\varepsilon}(x,y,T)\right\|_0\right)+$$

$$+\sqrt{ab}\left(\left\|u_2(x,y,T)\right\|_0+\left\|u_{2\varepsilon}(x,y,T)\right\|_0\right)\le 2m\left(|b|+\sqrt{ab}\right).$$

Using the above estimates and the result of Theorem 2, we obtain the following estimates:

$$\left\|U_{1\varepsilon}(x,y,t)\right\|_0\le\frac{2}{|b|}\left(\left(\varepsilon\left(|b|+\sqrt{ab}\right)\right)^{1-\frac{t}{T}}\left(2m\left(|b|+\sqrt{ab}\right)\right)^{\frac{t}{T}}\right),$$

$$\left\|U_{2\varepsilon}(x,y,t)\right\|_0\le\frac{2}{\sqrt{ab}}\left(\left(\varepsilon\left(|b|+\sqrt{ab}\right)\right)^{1-\frac{t}{T}}\left(2m\left(|b|+\sqrt{ab}\right)\right)^{\frac{t}{T}}\right).$$

Theorem 3 is proved.

**References:**


1. Tikhonov, A. N., Arsenin, V. Y. *Solutions of Ill-Posed Problems*. Winston & Sons, Washington, D.C., 1977.

2. Lavrentiev, M. M., Romanov, V. G., Shishatskii, S. P. *Ill-Posed Problems of Mathematical Physics and Analysis*. American Mathematical Society, Providence, RI, 1986.

3. Ivanov, V. K., Vasin, V. V., Tanana, V. P. *Theory of Linear Ill-Posed Problems and Its Applications*. VSP, Utrecht, 2002.

4. Kozhanov A. I. Well-posedness and ill-posedness of boundary-value problems for one class of fourth-order differential equations of Sobolev type // Itogi Nauki i Tekhniki. Sovremennaya Matematika i ee Prilozheniya. Tematicheskie Obzory. 2021. Vol. 198. P. 68–75

5. Pyatkov, S. G. *Some Classes of Ill-Posed Problems*. VSP, Utrecht, 2012.

6. Pyatkov, S. G. *Operator Theory. Nonclassical Problems*. VSP, Utrecht, 2002.

7. Tersenov, S. A. *Parabolic Equations with Changing Time Direction*. Novosibirsk: Nauka, 1985. *(rus tilida)*

8. Fayazov K. S., Khudaybergenov Ya. K. An ill-posed boundary value problem for a system of mixed-type equations with two degenerate planes, Siberian Electronic Mathematical Reports, Vol. 17, pp. 647–660 (2020). (in Russian)

9. Kerefov, A. A. *Mixed-Type Equations and Boundary Value Problems*. Vladikavkaz, 2000.

10. Egorov, I. E. "Boundary value problems for parabolic equations with degeneration." *Differential Equations*, 35 (1999), 1123–1131.

11. Khazhiev I. O. Conditional stability of the boundary value problem for a high-order system of mixed-type equations, Scientific Bulletin of SamSU, No. 1, pp. 60–67 (2022). (in Russian)
12. Fayazov, K.S., Khudayberganov, Y.K. Nonlocal boundary value problem for a nonhomogeneous parabolic type equation with two degenerate planes, Uzbek Mathematical Journal Open source preview, 2024, 68(3), Pp 53–65.
13. Fayazov, K.S., Khudayberganov, Y.K., Pyatkov S.G. Journal of Mathematical Sciences United StatesOpen source preview, 2023, 274(2), Pp 201–214.
14. Пятков С.Г., Неклассические дифференциальные уравнения в частных производных // Новосибирск, (1988), 77 с.
15. Пятков С.Г., Абашеева Н.Л., Разрешимость краевых задач для операторно-дифференциальных уравнений смешанного типа. Вырожденный случай, Сиб. матем. журн., 43(3) (2002), с. 678–693.
16. Крейн М.Г., Гохберг И. Ц. Введение в теорию линейных несамосопряженных операторов в гильбертовом пространстве // М.: Наука (1965). 448 с.
17.Рахмонов, З. Р., Ярметова, Д. И., & Исломов, Б. А. Автомодельное решение для задачи диффузии с нелинейными граничными условиями и источником в многомерном пространстве. Сборник Материалов Республиканской Научной Конференции «Актуальные Проблемы Прикладной Математики, Математического Моделирования и Информатики», (2024), с. 238-239.
18. Akbar oʻgli, I. B. (2024). Nodivergent Turidagi Oʻzgaruvchan Zichlikka Ega Kross-diffuziya Sistemasining Avtomodel Yechimi va Sonli Yaqinlashtirish. *IMRAS*, *7*(6), 531-536.