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BIRINCHI TARTIBLI ODDIY DIFFERENSIAL TENGLAMALARNI SONLI YECHISH USULLARI: EULER VA RUNGE–KUTTA YONDOSHUVLARI

Annotatsiya: Ushbu maqolada birinchi tartibli oddiy differensial tenglamalarni sonli yechish usullari qisqacha bayon etilgan. Euler usuli hamda ikkinchi tartibli Runge–Kutta usullari ularning keltirib chiqarilishi, aniqligi va yaqinlashuv xossalari nuqtai nazaridan koʻrib chiqilgan. Tadqiqot natijalari Runge–Kutta usullari Euler usuliga nisbatan yuqori aniqlikka ega ekanligini va amaliy masalalarni yechishda samarali ekanligini koʻrsatadi.

Kalit so'zlar: Differensial tenglamalar; Euler usuli; Runge–Kutta usullari; Sonli yechim; Yaqinlashuv; Kasr tartibli differensial tenglamalar

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NUMERICAL METHODS FOR SOLVING FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS: EULER AND RUNGE–KUTTA APPROACHES

Abstact: This paper presents a brief overview of numerical methods for solving first-order ordinary differential equations. Euler's method and second-order Runge–

Kutta methods are discussed, with emphasis on their derivation, accuracy, and convergence properties. The study shows that Runge–Kutta methods provide improved accuracy over Euler's method and are effective for practical scientific and engineering problems.
Key words: Differential equations; Euler's method; Runge–Kutta methods; Numerical solution; Convergence; Fractional-order differential equations

I. Definition

An equation that consists of derivatives is called a differential equation. Differential equations have applications in all areas of science and engineering. Mathematical formulation of most of the physical and engineering problems lead to differential equations. So, it is important for engineers and scientists to know how to set up differential equations and solve them.

Differential equations are of two types

1) ordinary differential equation (ODE)
2) partial differential equations (PDE).

An ordinary differential equation is that in which all the derivatives are with respect to a single independent variable. Examples of ordinary differential equation include

1) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$, $\frac{dy}{dx}(0)=2, y(0)=4$,

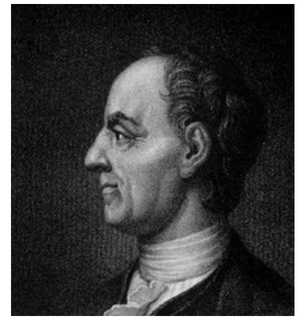
2) $\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}+5\frac{dy}{dx}+y=\sin x, \frac{d^2y}{dx^2}(0)=12, \frac{dy}{dx}(0)=2$, $y(0)=4$

Note: In this first part, we will see how to solve ODE of the form

$$\frac{dy}{dx}=f(x,y), y(0)=y_0$$

In another section, we will discuss how to solve higher order ordinary differential equations or coupled (simultaneous) differential equations.

II. Euler's Method



Swiss mathematician who was tutored by Johann Bernoulli. He worked at the Petersburg Academy and Berlin Academy of Science. He had a phenomenal memory, and once did a calculation in his head to settle an argument between students whose computations differed in the fiftieth decimal place. Euler lost sight in his right eye in 1735, and in his left eye in 1766. Nevertheless, aided by his phenomenal memory (and having practiced writing on a large slate when his sight was failing him), he continued to publish his results by dictating them. Euler was the most prolific mathematical writer of all times finding time (even with his 13 children) to publish over 800 papers in his lifetime. He won the Paris Academy Prize 12 times. When asked for an explanation why his memoirs flowed so easily in such huge quantities, Euler is reported to have replied that his pencil seemed to surpass him in intelligence. François Arago said of him "He calculated just as men breathe, as eagles sustain themselves in the air" (Beckmann 1971, p. 143; Boyer 1968, p. 482).

We will use Euler's method to solve an ODE under the form:

$$\frac{dy}{dx}=f(x,y),\ y(0)=y_0$$

At $x=0$, we are given the value of $y=y_0$. Let us call $x=0$ as x_0. Now since we know the slope of y with respect to x, that is, $f(x,y)$, then at $x=x_0$, the slope is $f(x_0,y_0)$. Both x_0 and y_0 are known from the initial condition $y(x_0)=y_0$.

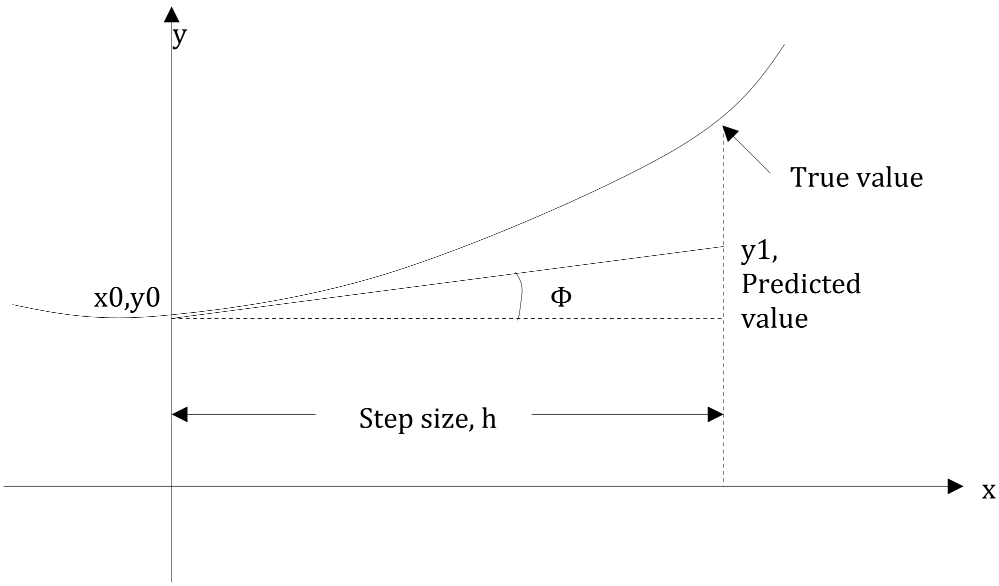


Figure 1. Graphical interpretation of the first step of Euler's method.

So the slope at $x=x_0$ as shown in the figure above

$$\text{Slope} = \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f(x_0, y_0)$$

Thus

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

If we consider $x_1 - x_0$ as a step size h, we get

$$y_1 = y_0 + f(x_0, y_0)h.$$

We are able now to use the value of y_1 (an approximate value of y at $x=x_1$) to calculate y_2, which is the predicted value at x_2,

$$y_2 = y_1 + f(x_1, y_1)h$$

$$x_2 = x_1 + h$$

Based on the above equations, if we now know the value of $y=y_i$ at x_i, then

$$y_{i+1}=y_i+f\left(x_i,y_i\right)h$$

This formula is known as the Euler's method and is illustrated graphically in Figure 2. In some books, it is also called the Euler-Cauchy method.

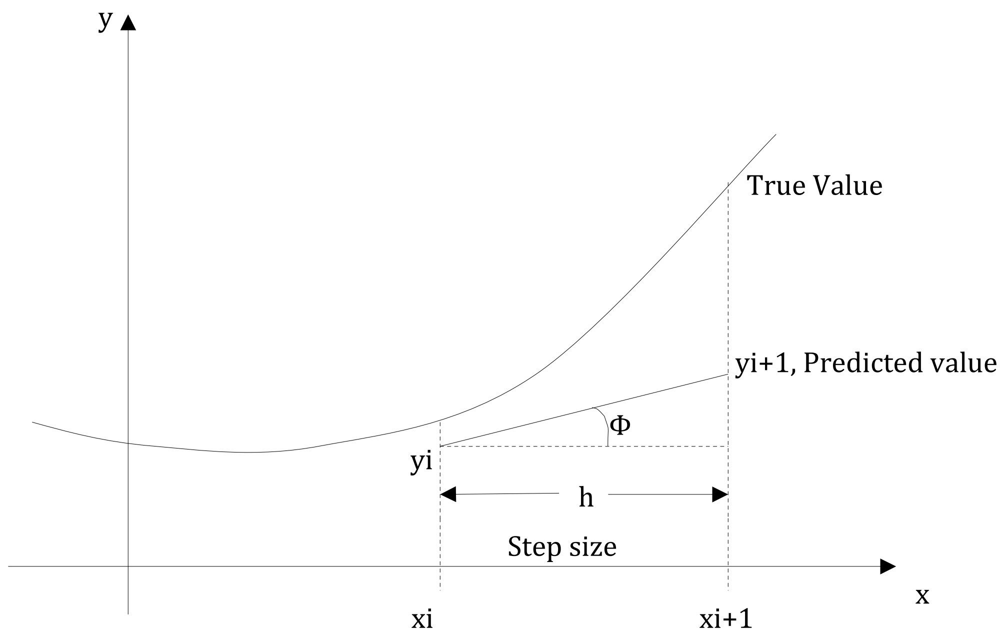


Figure 2. General graphical interpretation of Euler's method.

II. Runge-Kutta 2nd order

Euler's method was derived from Taylor series as:

$$y_{i+1}=y_i+f\left(x_i,y_i\right)h$$

This can be considered to be Runge-Kutta 1st order method.

The true error in the approximation is given by

$$E_t = \frac{f'(x_i, y_i)}{2!}h^2 + \frac{f''(x_i, y_i)}{3!}h^3 + \ldots$$

Now let us consider a 2nd order method formula. This new formula would include one more term of the Taylor series as follows:

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2$$

Let us now apply this to a simple example:

$$\frac{dy}{dx} = e^{-2x} - 3y,\ y(0) = 5$$

$$f(x, y) = e^{-2x} - 3y$$

Now since y is a function of x,

$$f'(x, y) = \frac{\partial f(x, y)}{\partial x} + \frac{\partial f(x, y)}{\partial y}\frac{dy}{dx}$$

$$= \frac{\partial}{\partial x}\left(e^{-2x} - 3y\right) + \frac{\partial}{\partial y}\left[\left(e^{-2x} - 3y\right)\right]\left(e^{-2x} - 3y\right)$$

$$= -2e^{-2x} + (-3)\left(e^{-2x} - 3y\right)$$

$$= -5e^{-2x} + 9y$$

The 2nd order formula would be

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2$$

$$y_{i+1} = y_i + \left(e^{-2x_i} - 3y_i\right)h + \frac{1}{2!}\left(-5e^{-2x_i} + 9y_i\right)h^2$$

You could easily notice the difficulty of having to find $f'(x, y)$ in the above method. What Runge and Kutta did was write the 2nd order method as

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

This form allows us to take advantage of the 2nd order method without having to calculate $f'(x, y)$.

But, how do we find the unknowns a_1, a_2, p_1 and q_{11}? Equating the above equations:

$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!} f'(x_i, y_i) h^2$ and $y_{i+1} = y_i + (a_1 k_1 + a_2 k_2) h$

gives three equations.

$$a_1 + a_2 = 1$$

$$a_2 p_1 = \frac{1}{2}$$

$$a_2 q_{11} = \frac{1}{2}$$

Since we have 3 equations and 4 unknowns, we can assume the value of one of the unknowns. The other three will then be determined from the three equations. Generally the value of a_2 is chosen to evaluate the other three constants. The three values generally used for a_2 are $\frac{1}{2}$, 1 and $\frac{2}{3}$, and are known as Heun's Method, Midpoint method and Ralston's method, respectively.

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